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An efficient difference scheme for the non-Fickian time-fractional diffusion equations with variable coefficient

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ABSTRACT

In this paper, we develop an efficient difference scheme for the non-Fickian time-fractional diffusion equations with variable coefficient. This model may be considered as a generalization of the Kolmogorov–Petrovskii–Piskunov type equation, which is widely used to describe some important phenomena in the fields of chemistry, biology and viscoelastic materials. The stability and convergence of the difference scheme in the maximum norm are proved by the discrete energy method under mild conditions. A numerical example is carried out to verify our theoretical analysis results.

1. Introduction

In this paper, we focus on the non-Fickian time-fractional diffusion equations in the following form

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,t)=\kappa_1 \frac{\partial}{\partial x}\left(\omega(x,t)\frac{\partial u(x,t)}{\partial x}\right)+\frac{\kappa_2}{\delta}\int_0^t e^{-\frac{t-s}{\delta}}\frac{\partial^2 u(x,s)}{\partial x^2}ds+f(x,t), 0<x<L, 0<t\le T, \tag{1.1}$$

$$ u(x,0)=\varphi(x),\ 0<x<L; \quad u(0,t)=u_0(t),\ u(L,t)=u_L(t),\ 0\le t\le T, \tag{1.2}$$

where δ,κ_1,κ_2 are positive constants, ${}_0^C\mathcal{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α defined by

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,\eta)}{\partial \eta}\frac{1}{(t-\eta)^{\alpha}}d\eta, \quad 0<\alpha<1,$$

and $\Gamma(\cdot)$ denotes the Euler's Gamma function. Also we suppose that there exist two constants C_1 and C_2 such that the given smooth function $\omega(x,t)$ satisfies $0<C_1\le\omega(x,t)\le C_2$ when $0\le x\le L, 0\le t\le T$.

This model is widely used to describe the phenomena of the wave propagation in the non-equilibrium media come from physics, chemistry, biology and so on, see e.g., [1]. The non-Fickian diffusion occurs also in viscoelastic materials where the classical diffusion equation has been replaced by a Volterra type equation, see

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e.g., [2,3] and references therein. Also, this model may be considered as a generalization of the Kolmogorov–Petrovskii–Piskunov (KPP) type equation, which is obtained by introducing the non Fickian flux to overcome the limitations of the classical KPP equations [4].

As is known to all, it is almost impossible to obtain a closed-form solution of fractional differential equations in general. As a result, the development of effective numerical methods becomes an important option. Up to now, almost all the existing works focus on the classical integer order problems. For instance, Li et al. [5] investigate the long time behavior of non-Fickian delay reaction–diffusion equations. Zhang et al. [6] construct two types of higher-order linearized multistep difference schemes for same problems. Ferreira et al. [7] developed some finite difference discretizations for the quasilinear non-Fickian diffusion equations of Volterra type with non singular and weakly singular kernels, where the convergence results are established in the sense of the discrete L^2 norm by using an unconventional means which can reduce the requirement of smoothness of theoretical solution when the usual split technique is used.

Compared with the classical integer order problems, two potential integral terms of the fractional problem (1.1)–(1.2) significantly increases the difficulty in algorithm construction and numerical analysis. Thus, it is necessary to construct efficient numerical approximations for such problem. Given all this, we propose an efficient difference scheme for the non-Fickian time-fractional problem (1.1)–(1.2) with variable coefficient by combining the popular L_1 formula and the compound trapezoidal formula. More importantly, the difference scheme is proved to be stable and convergent in the maximum norm under mild conditions.

The structure of this paper is organized as follows. In Section 2, we construct a difference scheme for the problem (1.1)–(1.2). In Section 3, a crucial priori estimate is first proved via the energy method, and the stability and convergence of the scheme in the maximum norm are studied. In Section 4, some numerical results are provided to verify our theoretical analysis. A brief conclusion is given in Section 5.

2. Derivation of the difference scheme

We first define a partition of the rectangle $[0,L]\times[0,T]$ by the mesh $\Omega_{h,\tau}=\Omega_h\times\Omega_\tau$, where $\Omega_h=\{x_i|x_i=ih, 0\le i\le M\}$, $\Omega_\tau=\{t_k|t_k=k\tau, 0\le k\le N\}$ with $h=L/M$ and $\tau=T/N$. Let $\mathcal{V}_h=\{v|v=(v_0,v_1,\ldots,v_M)\}$ be grid function space on Ω_h and $\mathring{\mathcal{V}}_h=\{v|v\in\mathcal{V}_h, v_0=v_M=0\}$.

For any $u,v\in\mathring{\mathcal{V}}_h$, we introduce the following discrete inner products

$$(u,v)=h\sum_{i=1}^{M-1}u_iv_i,\ \ (u,v)_\omega=h\sum_{i=1}^{M-1}u_iv_i\omega_i, \tag{2.1}$$

and norms and seminorms

$$\|u\|=\sqrt{(u,v)},\ \ |u|_1=\sqrt{(\delta_xu,\delta_xu)},\ \ \|u\|_\omega=\sqrt{(u,u)_\omega},\ \ |u|_{1,\omega}=\sqrt{(\delta_xu,\delta_xu)_\omega}, \tag{2.2}$$

where $\delta_xv_{i+\frac{1}{2}}=\frac{1}{h}(v_{i+1}-v_i)$. Also, we denote

$$D_\tau^\alpha u^k=\frac{1}{\Gamma(1-\alpha)\tau}\Big[a_0^{(\alpha)}u^k-\sum_{j=1}^{k-1}(a_{k-j-1}^{(\alpha)}-a_{k-j}^{(\alpha)})u^j-a_{k-1}^{(\alpha)}u^0\Big], \tag{2.3}$$

where $a_k^{(\alpha)}=\frac{\tau^{1-\alpha}}{1-\alpha}[(k+1)^{1-\alpha}-k^{1-\alpha}],\ k\in\mathbb{N}$. Then, we have the following lemma.

Lemma 2.1 ([8]). *Suppose that $f\in C^2[0,t_k]$ and $0<\alpha<1$. Let*

$$R^k(f)={}_0^C\mathcal{D}_t^\alpha f(t_k)-D_\tau^\alpha f^k, \tag{2.4}$$

then we have

$$|R^k(f)|\le\frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12}+\frac{2^{2-\alpha}}{2-\alpha}-(1+2^{-\alpha})\right]\max_{0\le t\le t_k}|f''(t)|. \tag{2.5}$$

Suppose $u \in C_{x,t}^{(4,2)}([0,L]\times[0,T])$, i.e., the solution $u(x,t)$ of the problem (1.1)–(1.2) has continuous partial derivatives up to the fourth order in space and second order in time. Define the grid functions

$$U_i^k = u(x_i,t_k),\ \omega_i^k = \omega(x_i,t_k),\ f_i^k = f(x_i,t_k),\quad 0\le i\le M, 0\le k\le N.$$

Let $g(x,t,s)=\rho(t,s)\frac{\partial^2 u(x,s)}{\partial x^2}$ with $\rho(t,s)=e^{-\frac{t-s}{\delta}}$, considering (1.1) at (x_i,t_k) we have

$$ {}_0^C\mathcal{D}_t^\alpha u(x_i,t_k) = \kappa_1 \frac{\partial}{\partial x}\left(\omega\frac{\partial u}{\partial x}\right)\Big|_{(x_i,\ t_k)} + \frac{\kappa_2}{\delta}\int_0^{t_k} g(x_i,t_k,s)ds + f(x_i,t_k), 1\le i\le M-1, 1\le k\le N. \tag{2.6}$$

Applying the composite trapezoidal rule, we obtain

$$\int_0^{t_k} g(x_i,t_k,s)ds = \sum_{l=0}^{k-1}\int_{t_l}^{t_{l+1}} g(x_i,t_k,s)ds = \frac{\tau}{2}\sum_{l=0}^{k-1}(\rho_k^l\delta_x^2U_i^l + \rho_k^{l+1}\delta_x^2U_i^{l+1}) + \mathcal{O}(h^2+\tau^2), \tag{2.7}$$

where $\rho_k^l=\rho(t_k,t_l)$ and $\delta_x^2U_i^l = \frac{1}{h}(\delta_xU_{i+\frac12}^l - \delta_x^2U_{i-\frac12}^l)$. This together with Lemma 2.1 and

$$\frac{\partial}{\partial x}\left(\omega\frac{\partial u}{\partial x}\right)\Big|_{(x_i,\ t_k)} = \delta_x(\omega_i^k\delta_xU_i^k) + \mathcal{O}(h^2), \tag{2.8}$$

we can obtain from (2.6) that

$$D_\tau^\alpha U_i^k = \kappa_1\delta_x(\omega_i^k\delta_xU_i^k) + \frac{\kappa_2\tau}{2\delta}\sum_{l=0}^{k-1}(\rho_k^l\delta_x^2U_i^l + \rho_k^{l+1}\delta_x^2U_i^{l+1}) + f_i^k + R_i^k, 1\le i\le M-1, 1\le k\le N, \tag{2.9}$$

and there exists a positive constant C_3 such that

$$|R_i^k| \le C_3(\tau^{2-\alpha}+h^2),\ 1\le i\le M-1, 1\le k\le N. \tag{2.10}$$

Omitting the small terms in (2.9), we obtain the difference scheme as follows

$$D_\tau^\alpha u_i^k = \kappa_1\delta_x(\omega_i^k\delta_xu_i^k) + \frac{\kappa_2\tau}{2\delta}\sum_{l=0}^{k-1}(\rho_k^l\delta_x^2u_i^l + \rho_k^{l+1}\delta_x^2u_i^{l+1}) + f_i^k,\ 1\le i\le M-1, 1\le k\le N, \tag{2.11}$$

$$u_i^0 = \varphi(x_i),\ 0\le i\le M, \tag{2.12}$$

$$u_0^k = u_0(t_k), u_M^k = u_L(t_k),\ 0\le k\le N, \tag{2.13}$$

where the initial boundary conditions (1.2) have been used.

3. Stability and convergence

In this section, we focus on analyzing the stability and convergence of the difference scheme (2.11)–(2.13). Before doing that, we introduce and prove several necessary lemmas.

Lemma 3.1 ([8]).*Suppose $\alpha\in(0,1)$, $a_k^{(\alpha)}(0\le k\le n-1, n\ge 1)$ is defined by (2.3), then it holds that*

$$a_0^{(\alpha)} > a_1^{(\alpha)} > a_2^{(\alpha)} > \cdots > a_{n-2}^{(\alpha)} > a_{n-1}^{(\alpha)}. \tag{3.1}$$

Lemma 3.2 ([9,10]). *For any grid function $v\in\mathring{\mathcal{V}}_h$, we have*

$$\|v\|_\infty^2 \le \frac{L}{4}|v|_1^2,\ \|v\|^2\le\frac{L^2}{6}|v|_1^2,\ C_1\|v\|^2\le\|v\|_\omega^2\le C_2\|v\|^2,\ C_1|v|_1^2\le|v|_{1,\omega}^2\le C_2|v|_1^2. \tag{3.2}$$

Next, we give a priori estimate, which plays a crucial role in the analysis of the difference scheme.

Theorem 3.3. *Suppose $\{v_i^k | 0 \le i \le M, 0 \le k \le N\}$ satisfies that*

$$D_\tau^\alpha v_i^k = \kappa_1 \delta_x(\omega_i^k \delta_x v_i^k) + \frac{\kappa_2 \tau}{2\delta} \sum_{l=0}^{k-1} (\rho_k^l \delta_x^2 v_i^l + \rho_k^{l+1} \delta_x^2 v_i^{l+1}) + q_i^k,\ 1 \le i \le M-1, 1 \le k \le N, \tag{3.3}$$

$$v_i^0 = \phi(x_i),\ 0 \le i \le M, \tag{3.4}$$

$$v_0^k = 0, v_M^k = 0,\ 0 \le k \le N. \tag{3.5}$$

Then we have

$$\tau \sum_{k=1}^{m} \|v^k\|_\infty^2 \le \frac{L}{4}\left(\frac{9\kappa_2^2 T^2}{9\kappa_2^2 T + 1} + \frac{T^{1-\alpha} L^2}{2\kappa_1 \Gamma(2-\alpha) C_1}\right) |\phi|_1^2 + \frac{3L^3}{16\kappa_1^2 C_1^2} \tau \sum_{k=1}^{m} \|q^k\|^2, \quad 1 \le m \le N, \tag{3.6}$$

when $\tau < \tau_0 := (2\kappa_1^2 \delta^2 C_1^2)/(9\kappa_2^2 T + 1)$.

Proof. Multiplying (3.3) by hv_i^k and summing up for i from 1 to $M-1$ and for k from 1 to m, we obtain

$$h \sum_{i=1}^{M-1} \sum_{k=1}^{m} (D_\tau^\alpha v_i^k) v_i^k = h \sum_{i=1}^{M-1} \sum_{k=1}^{m} \left[\kappa_1 (\delta_x(\omega_i^k \delta_x v_i^k)) v_i^k + \frac{\kappa_2 \tau}{2\delta} \sum_{l=0}^{k-1} (\rho_k^l \delta_x^2 v_i^l + \rho_k^{l+1} \delta_x^2 v_i^{l+1}) v_i^k + q_i^k v_i^k\right]. \tag{3.7}$$

Using the Cauchy–Schwarz inequality and noticing that $0 < \rho_{k+1}^l < \rho_{k+1}^{l+1} < 1$, we have

$$h \sum_{i=1}^{M-1} \sum_{k=1}^{m} (\delta_x(\omega_i^k \delta_x v_i^k)) v_i^k = -\sum_{k=1}^{m} |v^k|_{1,\omega}^2,\ h \sum_{i=1}^{M-1} \sum_{k=1}^{m} q_i^k v_i^k \le \frac{2\kappa_1 C_1}{L^2} \sum_{k=1}^{m} \|v^k\|^2 + \frac{L^2}{8\kappa_1 C_1} \sum_{k=1}^{m} \|q^k\|^2, \tag{3.8}$$

and

$$\frac{\kappa_2 \tau}{2\delta} h \sum_{i=1}^{M-1} \sum_{k=1}^{m} \sum_{l=0}^{k-1} (\rho_k^l \delta_x^2 v_i^l + \rho_k^{l+1} \delta_x^2 v_i^{l+1}) v_i^k \le C_4 \tau \sum_{k=1}^{m} \sum_{l=0}^{k-1} (|v^l|_1^2 + |v^{l+1}|_1^2) + C_5 \tau \sum_{k=1}^{m} |v^k|_1^2, \tag{3.9}$$

where $C_4 = \frac{3\kappa_1 \kappa_2^2 C_1}{2(9\kappa_2^2 T + 1)}$ and $C_5 = \frac{9\kappa_2^2 T + 1}{12\kappa_1 \delta^2 C_1}$. Moreover, we have

$$\begin{aligned}
\sum_{k=1}^{m} (D_\tau^\alpha v_i^k) v_i^k &= \frac{1}{\mu\tau} \sum_{k=1}^{m} \left[a_0^{(\alpha)} v_i^k - \sum_{j=1}^{k-1} (a_{k-j-1}^{(\alpha)} - a_{k-j}^{(\alpha)}) v_i^j - a_{k-1}^{(\alpha)} v_i^0\right] v_i^k \\
&\ge \frac{1}{\mu\tau}\left[\sum_{k=1}^{m} a_0^{(\alpha)} (v_i^k)^2 - \frac{1}{2}\sum_{k=2}^{m} (a_0^{(\alpha)} - a_{k-1}^{(\alpha)})(v_i^k)^2 - \frac{1}{2}\sum_{j=1}^{m-1} (a_0^{(\alpha)} - a_{m-j}^{(\alpha)})(v_i^j)^2 - \frac{1}{2}\sum_{k=1}^{m} a_{k-1}^{(\alpha)} [(v_i^0)^2 + (v_i^k)^2]\right] \\
&= \frac{1}{\mu\tau}\left[\frac{1}{2}\sum_{k=1}^{m} a_{m-k}^{(\alpha)} (v_i^k)^2 - \frac{1}{2}\sum_{k=1}^{m} a_{k-1}^{(\alpha)} (v_i^0)^2\right] \ge \frac{1}{\mu\tau}\left[\frac{1}{2} a_{m-1}^{(\alpha)} \sum_{k=1}^{m} (v_i^k)^2 - \frac{t_m^{1-\alpha}}{2(1-\alpha)} (v_i^0)^2\right],
\end{aligned} \tag{3.10}$$

where $\mu = \Gamma(1-\alpha)$ and the monotone property (3.1) in Lemma 3.1 has been used.

Substituting (3.8)–(3.10) into (3.7), and applying Lemma 3.2, we obtain

$$\begin{aligned}
&\frac{1}{\mu\tau}\left[\frac{1}{2}\sum_{k=1}^{m} a_{m-1}^{(\alpha)} \|v^k\|^2 - \frac{t_m^{1-\alpha}}{2(1-\alpha)} \|v^0\|^2\right] + \kappa_1 C_1 \sum_{k=1}^{m} |v^k|_1^2 \\
&\le C_4 \tau \sum_{k=1}^{m} \sum_{l=0}^{k-1} (|v^l|_1^2 + |v^{l+1}|_1^2) + C_5 \tau \sum_{k=1}^{m} |v^k|_1^2 + \frac{\kappa_1 C_1}{3} \sum_{k=1}^{m} |v^k|_1^2 + \frac{L^2}{8\kappa_1 C_1} \sum_{k=1}^{m} \|q^k\|^2.
\end{aligned} \tag{3.11}$$

Noticing that

$$\sum_{k=1}^{m}\sum_{l=0}^{k-1}(|v^l|_1^2+|v^{l+1}|_1^2)=m|v^0|_1^2+\sum_{k=1}^{m}(2(m-k)+1)|v^k|_1^2, \tag{3.12}$$

and $m\tau \le T$, we further have

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^{m}a_{m-1}^{(\alpha)}\|v^k\|^2+\frac{2\kappa_1C_1\mu}{3}\tau\sum_{k=1}^{m}|v^k|_1^2\\ &\le C_4T\mu\tau\Big(|v^0|_1^2+2\sum_{k=1}^{m}|v^k|_1^2\Big)+\frac{L^2\mu}{8\kappa_1C_1}\tau\sum_{k=1}^{m}\|q^k\|^2+\frac{t_m^{1-\alpha}}{2(1-\alpha)}\|v^0\|^2+C_5\mu\tau^2\sum_{k=1}^{m}|v^k|_1^2.\end{aligned} \tag{3.13}$$

This implies that

$$\left(1-\frac{3C_4T}{\kappa_1C_1}-\frac{3C_5\tau}{2\kappa_1C_1}\right)\tau\sum_{k=1}^{m}|v^k|_1^2\le C_6|v^0|_1^2+\frac{3L^2\tau}{16\kappa_1^2C_1^2}\sum_{k=1}^{m}\|q^k\|^2, \tag{3.14}$$

where

$$C_6=\frac{3C_4T^2}{2\kappa_1C_1}+\frac{T^{1-\alpha}L^2}{8\kappa_1C_1\Gamma(2-\alpha)}>\frac{3C_4T}{2\kappa_1C_1}\tau+\frac{T^{1-\alpha}L^2}{8\kappa_1C_1\Gamma(2-\alpha)}.$$

Noticing that

$$1-\frac{3C_4T}{\kappa_1C_1}-\frac{3C_5\tau}{2\kappa_1C_1}=1-\frac{9\kappa_2^2T}{2(9\kappa_2^2T+1)}-\frac{(9\kappa_2^2T+1)\tau}{8\kappa_1^2\delta^2C_1^2}>\frac{1}{2}-\frac{(9\kappa_2^2T+1)\tau}{8\kappa_1^2\delta^2C_1^2}>\frac{1}{4}$$

when $\tau<\tau_0$. Combining (3.3) and (3.14) with Lemma 3.2, we have

$$\tau\sum_{k=1}^{m}\|v^k\|_\infty^2\le\frac{L}{4}\left(\frac{9\kappa_2^2T^2}{9\kappa_2^2T+1}+\frac{T^{1-\alpha}L^2}{2\kappa_1\Gamma(2-\alpha)C_1}\right)|\phi|_1^2+\frac{3L^3}{16\kappa_1^2C_1^2}\tau\sum_{k=1}^{m}\|q^k\|^2,\quad 1\le m\le N. \tag{3.15}$$

This proof is completed. □

Based on above priori estimate, we can immediately obtain the following stability result.

Theorem 3.4 (*Stability*)**.** *The difference scheme (2.11)–(2.13) is stable in the maximum norm with respect to the initial value φ and the right hand side function f when $\tau<\tau_0$, where τ_0 is defined in (3.6).*

Next, we discuss the convergence of the difference scheme (2.11)–(2.13).

Theorem 3.5 (*Convergence*)**.** *Let $u\in C_{x,\,t}^{(4,2)}([0,L]\times[0,T])$ be solution of the problem (1.1)–(1.2), and u_i^k be solution of the difference scheme (2.11)–(2.13). Denote $e_i^k=u(x_i,t_k)-u_i^k$, then we have*

$$\tau\sum_{k=1}^{m}\|e^k\|_\infty^2\le\frac{3L^3T}{16\kappa_1^2C_1^2}C_3^2(\tau^{2-\alpha}+h^2)^2,\ 1\le m\le N. \tag{3.16}$$

Proof. From (2.9), we have the error equation as

$$D_\tau^\alpha e_i^k=\kappa_1\delta_x(\omega_i^k\delta_xe_i^k)+\frac{\kappa_2\tau}{2\delta}\sum_{l=0}^{k-1}(\rho_k^l\delta_x^2e_i^l+\rho_k^{l+1}\delta_x^2e_i^{l+1})+R_i^k,\ 1\le i\le M-1,1\le k\le N, \tag{3.17}$$

$$e_i^0=0,\ 0\le i\le M, \tag{3.18}$$

$$e_0^k=0,e_M^k=0,\ 0\le k\le N. \tag{3.19}$$

Applying the priori estimate in Theorem 3.3 and noticing that (2.10) and $|e^0|_1=0$, we get

$$\tau\sum_{k=1}^{m}\|e^k\|_\infty^2\le\frac{3L^3T}{16\kappa_1^2C_1^2}C_3^2(\tau^{2-\alpha}+h^2)^2,\ 1\le m\le N. \tag{3.20}$$

This proof is completed. □

Table 1
Errors and convergence orders for different α when $h = 1/1000$ (left) and $\tau = 1/10000$ (right).

τ	$\alpha = 0.30$		$\alpha = 0.70$		h	$\alpha = 0.30$		$\alpha = 0.70$	
	$E(h,\tau)$	Ord1	$E(h,\tau)$	Ord1		$E(h,\tau)$	Ord2	$E(h,\tau)$	Ord2
1/20	4.07e−4	–	3.36e−3	–	1/4	2.24e−3	–	2.26e−3	–
1/40	1.32e−4	1.63	1.39e−3	1.27	1/8	5.73e−4	1.97	5.77e−4	1.97
1/80	4.24e−5	1.63	5.74e−4	1.28	1/16	1.43e−4	2.00	1.45e−4	2.00
1/160	1.36e−5	1.64	2.35e−4	1.29	1/32	3.58e−5	1.99	3.67e−5	1.98
τ	$\alpha = 0.30$		$\alpha = 0.70$		h	$\alpha = 0.30$	h	$\alpha = 0.70$	
	$CE(h,\tau)$	COrd1	$CE(h,\tau)$	COrd1		$CE(h,\tau)$	COrd2	$CE(h,\tau)$	COrd2
1/20	3.54e−4	–	2.63e−3	–	1/4	1.11e−3	–	1.05e−3	–
1/40	1.10e−4	1.68	1.08e−3	1.29	1/8	2.77e−4	2.00	2.64e−4	1.99
1/80	3.44e−5	1.68	4.39e−4	1.29	1/16	6.97e−5	1.99	6.66e−5	1.98
1/160	1.07e−5	1.68	1.79e−4	1.30	1/32	1.74e−5	2.00	1.70e−5	1.97

4. Numerical results

In this section, we test the effectiveness of the method from two aspects of the maximum error $E(h,\tau) = \max_{1\le k\le N}\|U^k - u^k\|_\infty$ and the cumulative error $CE(h,\tau) = \sqrt{\tau\sum_{k=1}^{N}\|U^k - u^k\|_\infty^2}$. The convergence orders in time and space directions based on the maximum error are calculated by

$$\mathrm{Ord1} = \log_2\left(\frac{E(h,\tau)}{E(h,\tau/2)}\right), \mathrm{Ord2} = \log_2\left(\frac{E(h,\tau)}{E(h/2,\tau)}\right),$$

for sufficiently small h and τ respectively. The convergence order COrd1 and COrd2 based on the cumulative error are defined and calculated similarly.

Example 4.1. Consider the problem (1.1)–(1.2) with $\omega(x,t) = \sin^2(xt) + \frac{1}{2}$ on the domain $[0,1]\times[0,1]$. The functions $\varphi(x), u_0(t), u_L(t)$ and $f(x,t)$ are determined by the exact solution $u(x,t) = e^x t^2$.

In what follows, we apply the difference scheme (2.11)–(2.13) to solve the example for different values of α. Without loss of generality, we set the parameters $\kappa_1 = \kappa_2 = \sigma = 1$.

The errors and convergence orders for $\alpha = 0.30$ and 0.70 are shown in Table 1. We can observe that the index Ord1 and COrd1 always tend to $2-\alpha$, while Ord2 and COrd2 tend to 2 as τ and h are reduced by a factor of 2 respectively. It means that the suggested difference scheme has accuracy of order $2-\alpha$ in time and order 2 in space for $0 < \alpha < 1$, which is consistent with our theoretical analysis in Theorem 3.5.

5. Conclusion

In this paper, we propose an efficient difference scheme for the variable coefficient non-Fickian time-fractional diffusion equations. The stability and convergence under very mild conditions is shown by using the discrete energy method. The results in this paper can be directly extended to the fractional quasilinear non-Fickian equation [7] when the kernel to be governed by some positive constant. In future work, we hope to construct the high order algorithms for the fractional quasilinear non-Fickian diffusion problem.

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